Generic criticality in a model of evolution

Adam Lipowski*
Department of Physics, Adam Mickiewicz University, ulica Umultowska 85, 61-614 Poznań, Poland
(Received 30 March 2000; revised manuscript received 15 May 2000)

Using Monte Carlo simulations, we show that for a certain model of biological evolution, which is driven by nonextremal dynamics, active and absorbing phases are separated by a critical phase. In this phase both the density of active sites $\rho(t)$ and the survival probability of spreading $P(t)$ decay as $t^{-\delta}$, where $\delta \sim 0.5$. At the critical point that separates the active and critical phases $\delta \sim 0.29$, which suggests that this point belongs to the so-called parity-conserving universality class. Such a classification is also supported by finite-size analysis. The model has infinitely many absorbing states and, except for a single point, has no apparent conservation law.

PACS number(s): 05.70.Ln

I. INTRODUCTION

Recently, the statistical mechanics of complex systems has attracted a lot of attention. The main motivation for studying these systems is the belief that a qualitative understanding can be obtained by studying relatively simple mathematical models. Indeed, there are a number of examples where (spin) glasses, proteins, biological evolution, societies, or economies have been described in terms of very simple models [1,2]. In quantitative terms, complexity is very often related to the absence of a characteristic length or time scale (i.e., scale invariance). For example, paleontological data suggest that outbreaks of evolutionary activity substantially varied in size and in addition were correlated over large periods of time [3]. Similar features seem to characterize other, at first sight unrelated, processes like earthquakes, stock-exchange fluctuations, or the flow of sand.

The absence of characteristic scales is a well-known property of critical systems in the field of equilibrium statistical mechanics. However, in equilibrium statistical mechanics, criticality is an exception rather than a rule and it requires fine tuning of control parameter(s). On the other hand, the apparent abundance of scale invariance among complex systems suggests that this property should be in some sense generic and should not require such fine tuning.

An interesting idea that tries to explain the scale invariance in various systems was proposed by Bak, Tang, and Wiesenfeld under the name of self-organized criticality (SOC) [4]. They have shown that the dynamics of some simple systems might naturally lead these systems to a critical state. Such behavior was subsequently observed in a number of other models.

However, models exhibiting SOC are usually driven by very special dynamics. This is either the so-called extremal dynamics [5] (which drives some evolutionary models) or conservative dynamics (which drives sandpilelike models). When these dynamical rules are even slightly violated, the criticality is usually destroyed. For example, in the Bak-Sneppen model [6] describing the evolution of an ecosystem, the criticality is lost when we modify the rule that only the least fit species dies out and is replaced by a new one. However, on biological grounds, one expects that extinction might happen to a more fit species as well. It would be desirable to construct a model that would not be driven by such a special dynamics but whose criticality would be in some sense generic.

In search of generic criticality, we might recall that such models exist in equilibrium statistical mechanics; the prime example is the XY model. In this model the low-temperature phase is critical and correlation functions decay algebraically. Above a certain temperature the criticality of the model is destroyed and the system is in a disordered phase, where correlation functions decay exponentially [7].

Despite the wealth of models with critical behavior, the existence of generic criticality in nonequilibrium statistical mechanics is still an open problem [8]. In some cases, certain symmetries, conservation laws, or separation of time scales are responsible for the criticality of the system. These factors also play an important role in SOC models. Recently, some SOC models have been related to more general models [9–11]. It turns out that in some cases SOC corresponds to the critical points of these more general models. However, the criticality of the latter models most likely resembles ordinary criticality (i.e., the criticality exists only at some isolated points).

In the present paper we study a model of biological evolution. The model describes an ecosystem at the coarse-grained level similarly to the Bak-Sneppen model, but it is driven by nonextremal dynamics. We show that in some range of a control parameter certain quantities exhibit power-law behavior and thus the model might be said to be generically critical. The behavior of our model is determined by a certain symmetry, which places the model in the so-called parity-conserving (PC) universality class.

II. THE MODEL AND ITS PROPERTIES

A. Definition

Our model is a variant of other recently introduced models [12–14]. It is defined on a one-dimensional lattice, where for each bond between the nearest-neighbor sites i and $i+1$, we introduce bond variables $w_{i,i+1} \in (-0.5,0.5)$. Introducing the parameter r, we call the site i active when

*Email address: lipowski@main.amu.edu.pl

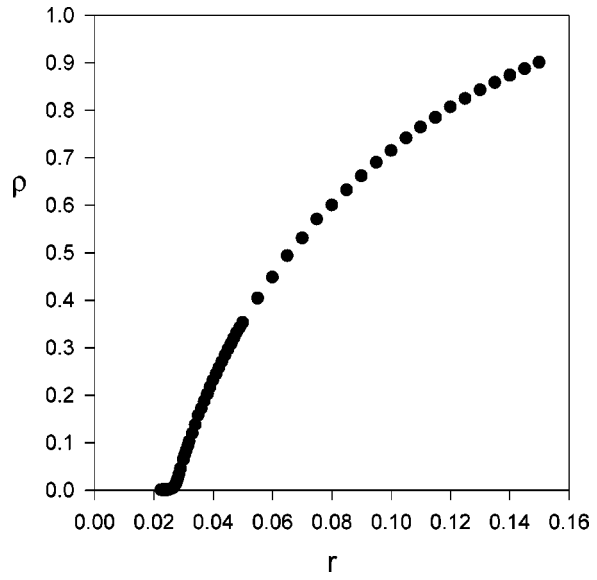

FIG. 1. The steady-state density of active sites ρ as a function of r.

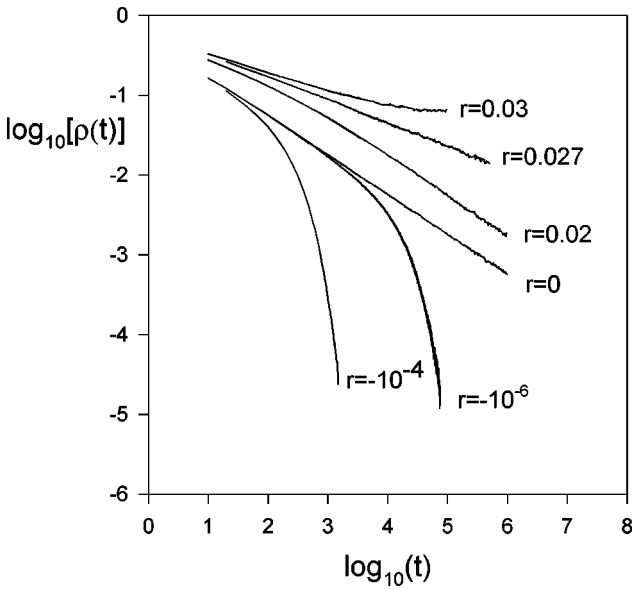

FIG. 2. The density of active sites $\rho(t)$ as a function of t plotted on a log-log scale. Simulations were made for $L=10^5$ and for each r we averaged over about 50 independent runs.

$$w_{i,i+1}w_{i-1,i}<r. \tag{1}$$

Otherwise, the site is called nonactive. The model is driven by random sequential dynamics and when the active site i is selected, we assign anew, with uniform probability, two bond variables $w_{i,i+1}$ and $w_{i-1,i}$. Nonactive sites are not updated, but updating a certain (active) site might change the status of its neighbors. The above rules immediately imply the existence of an absorbing state, i.e., one without active sites.

One can interpret the sites as species with the fitness being a product of the attached bond variables. When the fitness of a certain species is lower than a threshold value r, the species becomes extinct and is replaced by another species.

B. Steady-state properties

To examine the properties of our model, we used Monte Carlo simulations. Since the implementation of the above dynamical rules on a computer is straightforward, below we present only the results of these simulations. An important quantity characterizing this model is the steady-state density of active sites ρ. Figure 1 shows the density ρ as a function of r. The simulations were performed for the linear system size $L=10^5$ and we checked that the results presented are, within a small statistical error, size independent. For each r, after relaxing the random initial configuration for $t_{\rm rel}=10^4$, we made measurements during runs of $t=10^5$ (the unit of time is defined as a single update per lattice site on average). This figure suggests that the model undergoes a continuous transition at $r=r_c\sim 0.027$ and for $r<r_c$ the model should be in the absorbing phase with $\rho=0$. The simulations close to the critical point ($r<0.03$) were more extensive and we used $L=5\times 10^5$, $t_{\rm rel}=5\times 10^4$, and $t=10^6$. Let us also note that since bond variables are continuous there is a continuous degeneracy of the absorbing state.

C. Dynamical behavior

We observed that for $r<r_c$ the system is rather reluctant to reach an absorbing state. Only for $r<0$ does the system reach such a state quite quickly. The results of our simulations presented below show that for $0<r<r_c$ the model remains in the critical phase.

First, we examined the time evolution of the density of active sites $\rho(t)$. After the system starts from an arbitrary initial configuration containing a finite fraction of active sites, the density $\rho(t)$ should decay to zero in the absorbing phase, and this decay should be faster than a power law. At the critical point one expects a power-law decay $\rho(t)\sim t^{-\delta}$. In the active phase $\rho(t)$ asymptotically remains positive.

In Fig. 2 we show $\rho(t)$ as a function of t plotted on a logarithmic scale. For $r=0.03$ the density $\rho(t)$ clearly approaches a positive value, which confirms that in this case the model is in the active phase. Moreover, a faster than power-law decay is observed for $r=-10^{-4}$ and -10^{-6}. However, for $r=0$ and 0.02 our simulations show a power-law decay with the exponent $\delta=0.50(1)$. For $r=0.027$, which is very close to the critical point (see Fig. 1), the exponent $\delta=0.29(1)$.

Critical properties of models with absorbing states can be also studied using the so-called dynamic (or epidemic) method [15]. In this method, we prepare the system in one of the absorbing states except for a localized activity (usually at a single site). Subsequently, the system evolves according to its dynamical rules and and we monitor statistical properties of such runs. One of the important quantities in this technique is the probability $P(t)$ that a given activity survives until time t. On general grounds, one expects that $P(t)$ behaves similarly to $\rho(t)$, namely, in the active phase $P(t)$ tends to a finite value, in the absorbing phase it rapidly (faster than a power law) decreases to zero, and in the critical phase it decays as $t^{-\delta'}$, where δ' is an exponent that in general might be different from δ [16,17].

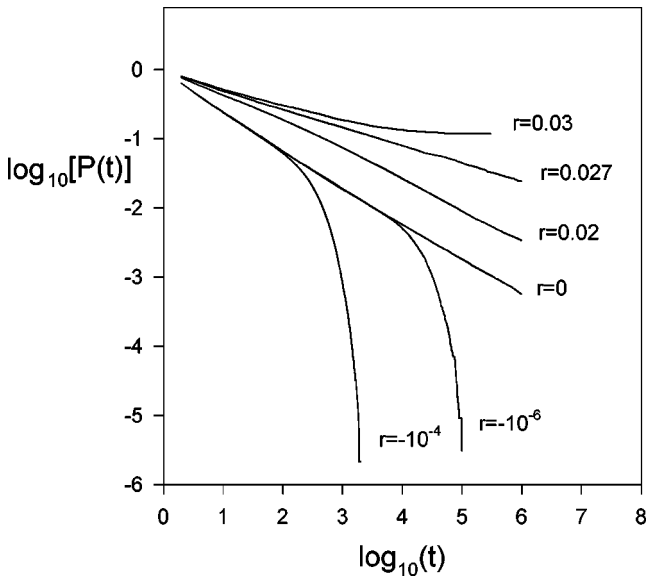

FIG. 3. The survival probability $P(t)$ as a function of t plotted on a log-log scale. Initially we set $w_{i,i+1}=w_0=0.2$ for all bonds except for those surrounding a certain site that was set as active. A different choice of w_0 might change the asymptotic slope at $r=r_c \sim 0.027$, but it should not affect the asymptotic $t^{-0.5}$ decay for $0 \leqslant r<r_c$.

Figure 3 shows the results of our dynamical simulations. Simulations were performed for sizes that ensured that the spreading activity did not reach the boundaries of the lattice (typically $L=2\times10^4$ is sufficient). The number of runs varied from 10^6 for $r=-10^{-4},-10^{-6}$, and 0 to 2×10^4 for $r=0.03$. The results shown in Fig. 3 lead to the same conclusions as those in Fig. 2: for $0\leqslant r<r_c$ the model remains in the critical phase with $\delta'=0.50(1)$. For $r<0$ the survival probability decays most likely faster than an inverse power of t.

The fact that for $r<0$ the model is in the absorbing phase is to some extent understood. That is, for $r<0$ there exists a finite probability that after updating a pair of sites will become nonactive forever. Indeed, when one of the newly selected bonds (say, $w_{i,i+1}$) satisfies the condition

$$|w_{i,i+1}|<-r/0.5, \tag{2}$$

then the sites i and $i+1$ become permanently nonactive. That is, no matter what other bonds (i.e., $w_{i-1,i}$ and $w_{i+1,i+2}$) are attached to these sites, they will always remain nonactive. For $r<0$ there is a finite probability of satisfying Eq. (2) and the above mechanism leads to the rapid decrease of active sites and hence the system quickly reaches an absorbing state. The above mechanism is not effective for $r \geqslant 0$ since there is no value that would ensure permanent nonactivity of a certain site. Essentially the same mechanism is at work in another model with absorbing states [14].

A more detailed analysis of ρ in the vicinity of the critical point suggests that the exponent β is slightly less then unity. Together with the value of $\delta\sim0.29$ at the critical point $r=r_c\sim0.027$ (as estimated from the results in Fig. 2), this strongly suggests that this model behaves similarly to some other models, which are commonly termed the parity-conserving universality class [18,19]. In this class of models,

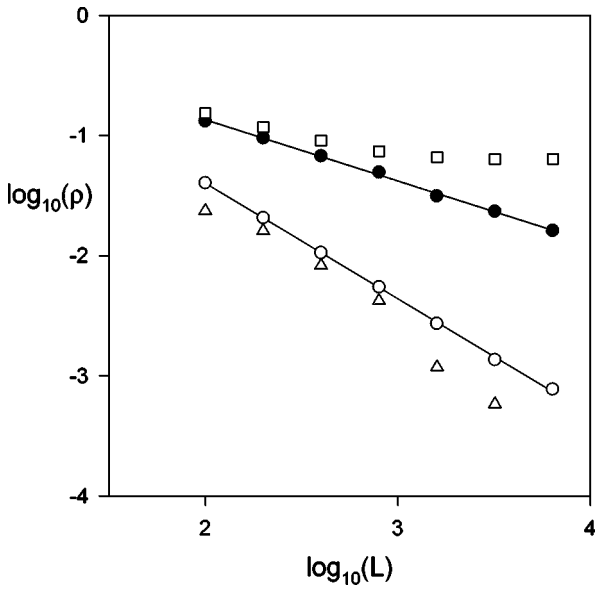

FIG. 4. The steady-state density ρ as a function of the system size L for $r=-10^{-4}$ (△), 0 (○), 0.027 (●), and 0.03 (□). For $r=0$ and 0.027 the estimated slope equals 0.98(3) and 0.49(3), respectively.

it is already known that in the critical phase $\delta=0.5$. An interesting feature of our model is the fact that the critical phase terminates at a certain point ($r=0$) and (most likely) an exponential decay sets in. Let us also notice that for $r=0.027$ the slope in Fig. 3 is also close to 0.29. However, this might be a coincidence, since some variability of this exponent in the dynamic Monte Carlo method is an anticipated feature [16,17].

Initially, the behavior of models belonging to the PC universality class was thought to be determined by the local conservation laws [18]. Later, however, some models were found that do not possess this property but that exhibit the

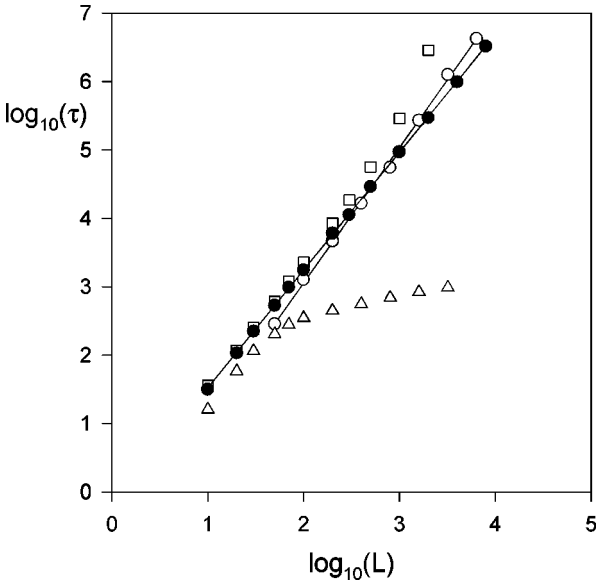

FIG. 5. The absorption time τ as a function of the system size L for $r=-10^{-4}$ (△), 0 (○), 0.027 (●), and 0.03 (□). For $r=0$ and 0.027 the estimated slope equals 1.98(5) and 1.73(5), respectively.

same critical behavior [19]. Since these models have a symmetric and double-degenerate absorbing state, it seems that this is another property that leads to PC criticality. Recently, a model with infinitely many absorbing states was found that also belongs to the PC universality class, but this might again be attributed to some conservation law in its dynamics [17]. The present model has an infinitely degenerate absorbing state and no conservation law [20]. We attribute the PC criticality of this model to a certain global symmetry of this model, namely, one can easily see that by inverting all bond variables ($w_{i,i+1} \rightarrow -w_{i,i+1}$) one does not change the (non)activity of any site. This Ising-like symmetry determines the structure of any absorbing state for $r \geqslant 0$: all bond variables must be either positive or negative.

Let us note that, although there is no conservation law in this model in general, there is such a law for $r=0$. Indeed, for $r=0$ the dynamics of the model is special: it is only the sign of $w_{i,i+1}$ that matters and only those sites are active where negative and positive $w_{i,i+1}$ meet. As a result, the dynamics of the model is equivalent to a particular branching annihilating random walk (BARW) with an even number of offspring [21]. The power-law characteristics $t^{-0.5}$ are already known for related BARW models [22].

D. Finite-size analysis

To provide yet another confirmation that our model belongs to the PC universality class we present the results of a finite-size analysis. In Fig. 4 we present the steady-state density ρ as a function of the system size L. One expects that at criticality $\rho \sim L^{-\beta/\nu_\perp}$ and for the PC universality class we have $\beta/\nu_\perp \sim 0.50(1)$. Moreover, in the critical phase ($0<r<r_c$) the density ρ should scale as L^{-1}. Such a behavior is clearly seen in Fig. 4. One can also see that in the active phase ρ converges to a positive value and in the absorbing phase ($r<0$) ρ decays most likely faster than a power of L.

A similar analysis for the average absorption time τ is shown in Fig. 5. At criticality τ increases as $L^{1.73}$ and such a behavior was also observed for some other models of the PC universality class. In the critical phase τ seems to increase as L^2. Moreover, in the active phase τ most likely increases faster than a power of L and in the absorbing phase slower than a power of L.

III. SUMMARY

In summary, we have shown that a particular evolutionary model with nonextremal dynamics exhibits generic criticality. Such a behavior is most likely related to a special symmetry of this model. One might hope that this criticality is to some extent robust with respect to structural perturbations of this model (other lattices, definitions of fitness function, etc.). Since for some of these variants the analogy with random walk models might not hold, it is possible that the criticality of such models will exhibit some sort of nonuniversality. However, analysis of such extensions is left as a future problem.

ACKNOWLEDGMENTS

I thank Dr. H. Hinrichsen for interesting discussion and the Department of Mathematics of Heriot-Watt University (Edinburgh, Scotland) for allocation of computer time.

[1] M. Mézard, G. Parisi, and M. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
[2] P. Bak, *How Nature Works* (Copernicus, New York, 1996).
[3] R. V. Sole, S. C. Manrubia, M. Benton, and P. Bak, Nature (London) **388**, 764 (1997).
[4] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[5] M. Paczuski, S. Maslov, and P. Bak, Phys. Rev. E **53**, 414 (1996).
[6] P. Bak and K. Sneppen, Phys. Rev. Lett. **71**, 4083 (1993).
[7] J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B **16**, 1217 (1977).
[8] G. Grinstein, in *Scale Invariance, Interfaces and Non-Equilibrium Dynamics*, edited by A. McKane *et al.* (Plenum Press, New York, 1995).
[9] D. Sornette, A. Johansen, and I. Dornic, J. Phys. I **5**, 325 (1995).
[10] A. Vespignani, R. Dickman, M. A. Munoz, and S. Zapperi, Phys. Rev. Lett. **81**, 5676 (1998).
[11] A. Vespignani, R. Dickman, M. A. Munoz, and S. Zapperi, e-print cond-mat/0003285.
[12] A. Lipowski and M. Łopata, Phys. Rev. E **60**, 1516 (1999).
[13] A. Lipowski, Phys. Rev. E **60**, R6255 (1999).
[14] A. Lipowski, Phys. Rev. E **62**, 4401 (2000).
[15] P. Grassberger and A. de la Torre, Ann. Phys. (N.Y.) **122**, 373 (1979).
[16] J. F. F. Mendes, R. Dickman, M. Henkel, and M. C. Marques, J. Phys. A **27**, 3019 (1994).
[17] M. C. Marques and J. F. F. Mendes, Eur. Phys. J. B **12**, 123 (1999).
[18] P. Grassberger, F. Krause, and T. von der Twer, J. Phys. A **17**, L105 (1984).
[19] M. H. Kim and H. Park, Phys. Rev. Lett. **73**, 2579 (1994); A. Lipowski, J. Phys. A **29**, L355 (1996); H. Hinrichsen, Phys. Rev. E **55**, 219 (1997); E. Carlon, M. Henkel, and U. Schollwöck, e-print cond-mat/9912347.
[20] For another model with infinitely many absorbing states and Ising-like symmetry, see W. Hwang and H. Park, Phys. Rev. E **59**, 4683 (1999).
[21] One can easily see that for $r=0$ the number of active sites in the system after updating an active site changes by 0 or 2.
[22] J. Cardy and U. C. Taüber, Phys. Rev. Lett. **77**, 4780 (1996); H. Takayasu and N. Inui, J. Phys. A **25**, L585 (1992).